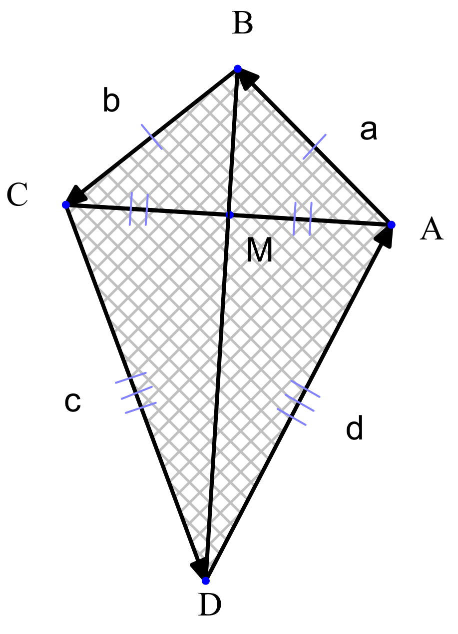
Vector Proof for the following quadrilateral properties

Shape	Definition	Properties of Diagonals		
		Equal in length	Perpen-dicular	Bisect each other
Trapezium	2 sides parallel	no	no	no
Kite	2 pairs of adjacent sides equal in length	no	yes	no
Parallelogram	Opposite sides parallel and equal in length	no	no	yes
Rectangle	Opposite sides parallel and equal in length. All angles 90	yes	no	yes
Rhombus	All sides equal in length	no	yes	yes
Square	Rectangle with all sides equal in length	yes	yes	yes

1. Kite : Proof diagonals are perpendicular



Let M be the midpoint of AC

$\overrightarrow{AC} = (\vec{a} + \vec{b})$

$\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB}$

$= \frac{1}{2}\overrightarrow{CA} + \vec{a}$

$= \frac{-1}{2}(\vec{a} + \vec{b}) + \vec{a}$

$= \frac{1}{2}(\vec{a} - \vec{b})$

$\overrightarrow{AC} \cdot \overrightarrow{MB} = \frac{1}{2}(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$

$= \frac{1}{2}(|a|^2 - |b|^2) = 0$

$\Rightarrow \overrightarrow{AC} \perp \overrightarrow{MB}$

$\overrightarrow{MD} = \overrightarrow{MA} + \overrightarrow{AD} = \frac{1}{2}(\vec{c} + \vec{d}) - \vec{d}$

$= \frac{1}{2}(\vec{c} - \vec{d})$

$\overrightarrow{MD} \cdot \overrightarrow{AC} = \frac{-1}{2}(\vec{c} - \vec{d}) \cdot (\vec{d} + \vec{c})$

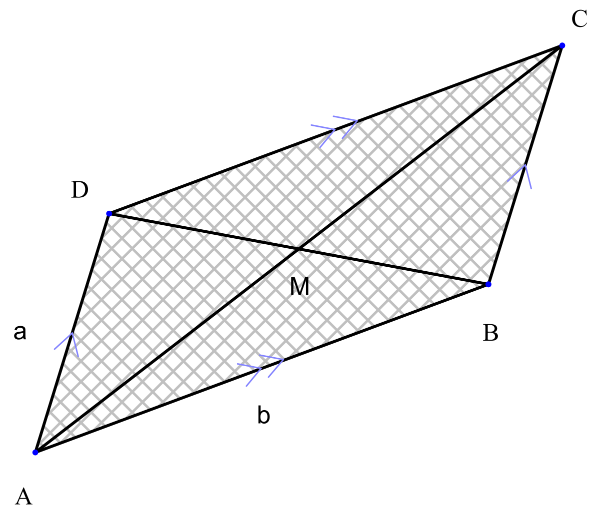
$= \frac{-1}{2}\left(|c|^2 - |d|^2\right) = 0$

$\Rightarrow MD \perp AC$

$\therefore$ BMD is a straight line

$\Rightarrow$ diagonals $\perp$ each other

2. Proof diagonals of parallelogram bisects each other



Let $\overrightarrow{DM} = x\overrightarrow{DB}; \overrightarrow{AM} = y\overrightarrow{AC}$

$\overrightarrow{DB} = \vec{b} - \vec{a}$; $\overrightarrow{AC} = \vec{b} + \vec{a}$

ΔADM :

$\vec{a} + \overrightarrow{DM} = \overrightarrow{AM}$

$\Rightarrow \vec{a} + x(\vec{b} - \vec{a}) = y(\vec{b} + \vec{a})$

$\Rightarrow \vec{b}(x - y) + \vec{a}(1 - x - y) = \vec{0}$

since $\vec{a}, \vec{b}$ independent,

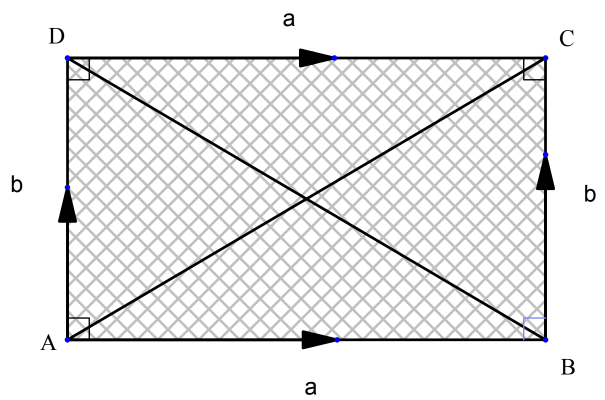
$\Rightarrow x - y = 0$; $1 - x - y = 0$

$\Rightarrow x = y$; $1 = 2x$

$\Rightarrow x = y = \dfrac{1}{2}$

Hence the diagonals bisects each other.

3. Rectangle : It is a parallelogram, hence diagonals bisects each other (see (2) above).
   Proof : diagonals equal in length



$\overrightarrow{AC} = \vec{a} + \vec{b}$

$\overrightarrow{DB} = \vec{a} - \vec{b}$

$|AC|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$\quad = |a|^2 + |b|^2 + 2a \cdot b$

$|DB|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

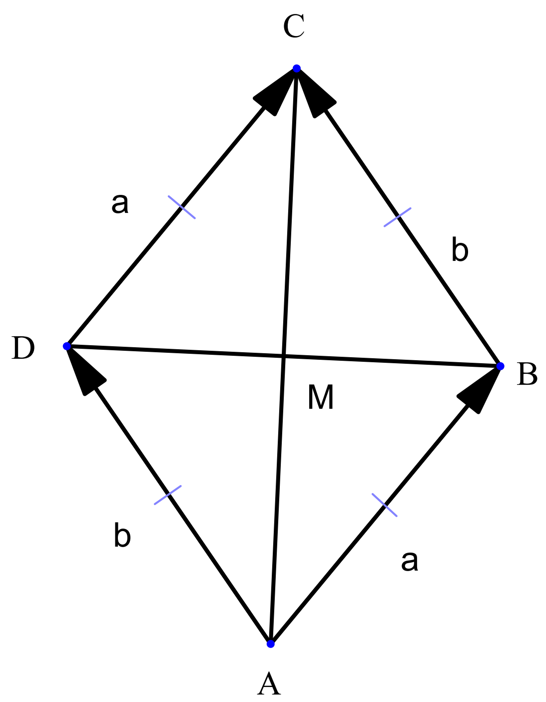
$\quad = |a|^2 + |b|^2 - 2a \cdot b$

Since $\vec{a} \cdot \vec{b} = 0$ $(\vec{a} \perp \vec{b})$

$\Rightarrow |AC|^2 = |DB|^2$

$\therefore |AC| = |DB|$

4. Rhombus : It is a parallelogram, hence diagonals bisects each other (see (2) above). Proof : diagonals perpendicular to each other



$$\overrightarrow{AC} = (\vec{b} + \vec{a})$$
$$\overrightarrow{BD} = \vec{b} - \vec{a}$$
$$\overrightarrow{AC} \cdot \overrightarrow{BD} = (\vec{b} + \vec{a}) \cdot (\vec{b} - \vec{a})$$
$$= |b|^2 - |a|^2 = 0$$
$$\Rightarrow AC \perp BD$$

5. Square : Since it is both a rhombus and a rectangle, the properties of the diagonals follow.