Transformations

Dilation

1.1 Dilation by a factor of d_x from x axis

- any point not on the x axis move vertically away from the x axis if $d_x > 1$
- any point not on the x axis move vertically towards the x axis if $d_x < 1$
- any point on the x axis stays on the x axis

Image $(x, y) \to (x, d_x y)$

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & d_x \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$y_1 = d_x y \Rightarrow y = \dfrac{y_1}{d_x}$

$x_1 = x$ becomes

$\dfrac{y_1}{d_x} = y \Rightarrow y_1 = d_x y$

dropping the suffix "1" for x_1 :

$$\boxed{y_1(x) = d_x \cdot y(x)} \qquad (1.1)$$

e.g. $y = x^2 \to y = 2x^2$

For $|d_x| > 1$ graph is taller. For $|d_x| < 1$ graph is fatter.

1.2 Dilation by a factor of d_y from y axis

- any point not on the y axis move horizontally away from the y axis if $d_y > 1$
- any point not on the y axis move horizontally towards the y axis if $d_y < 1$
- any point on the y axis stays on the y axis

Image $(x, y) \to (d_y x, y)$

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} d_y & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$x_1 = d_y x \Rightarrow x = \dfrac{x_1}{d_y}$

$y_1 = y$ becomes

$y_1(x) = y(x) = y\left(\dfrac{x_1}{d_y}\right)$

dropping the "1" for x_1:

$$\boxed{y_1(x) = y\left(\frac{x}{d_y}\right)} \qquad (1.2)$$

e.g. $y = x^2 \to y = \left(\dfrac{x}{2}\right)^2$

For $|a| > 1$ graph is expanded. For $|a| < 1$ graph is compressed.

2. Reflection

2.1 Reflection from x axis

Image (x,y) → (x,-y)

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\boxed{y_1 = -y(x)} \tag{1.3}$$

e.g. $y = x^2 \rightarrow y = -x^2$

2.2 Reflection from y axis

Image (x,y) → (-x,y)

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\boxed{y_1 = y(-x)} \tag{1.4}$$

e.g. $y = x^2 \rightarrow y = (-x)^2$

3. Translation of h unit in positive x direction, k unit in positive y direction

Image (x,y) → (x+h,y+k)

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$

$x_1 = x + h \Rightarrow x = x_1 - h$

$y_1 = y + k \Rightarrow y = y_1 - k$

Equation y = y(x) → $y_1 - k = y(x_1 - h) \Rightarrow y_1 = y(x_1 - h) + k$

Dropping the "1" for x_1 :

$$\boxed{y_1(x) = y(x-h) + k} \tag{1.5}$$

e.g. $y = x^2 \rightarrow y = (x-2)^2 + 3$ h=2, k=3

4. Dilation by a factor d_y from y axis, (reflection from y axis), dilation by a factor of d_x from x axis, (reflection from x axis), THEN translation h unit in positive x direction, k unit in positive y direction

$d_y < 0$ if reflection from y axis, $d_x < 0$ if reflection from x axis.

Matrix $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} h \\ k \end{bmatrix} \right) = \frac{1}{d_x d_y} \begin{bmatrix} d_x & 0 \\ 0 & d_y \end{bmatrix} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} h \\ k \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{d_y}(x_1 - h) \\ \frac{1}{d_x}(y_1 - k) \end{bmatrix}$$

$y_1(x_1) = d_x y(x) + k$

Image $(x_1, y_1) = (d_y x + h, d_x y + k)$

dropping the "1" in x_1 :

$$y_1(x) = d_x y\left(\frac{1}{d_y}(x - h) \right) + k \qquad (1.6)$$

Deviations from D-R-T

5. TDR/TRD

$$\begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} h \\ k \end{bmatrix}\right)=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Image $(x_1, y_1) = \left(d_y(x+h), d_x(y+k)\right)$

$$y_1(x) = d_x\left[y\left(\frac{1}{d_y}(x-d_y h)\right)+k \right] \tag{1.7}$$

6. DTR

$$\begin{bmatrix} r_y & 0 \\ 0 & r_x \end{bmatrix}\left(\begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} h \\ k \end{bmatrix}\right)=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

where $r_y = -1$ if reflection from y axis, otherwise $r_y = 1$, $r_x = -1$ if reflection from x axis, otherwise $r_x = 1$

Image $(x_1, y_1) = \left(r_y(d_y x+h), r_x(d_x y+k)\right)$

$$y_1(x) = r_x\left[d_x y\left(\frac{1}{r_y d_y}(x-r_y h)\right)+k \right] \tag{1.8}$$

7. RTD

$$\begin{bmatrix} d_y & 0 \\ 0 & d_x \end{bmatrix}\left(\begin{bmatrix} r_y & 0 \\ 0 & r_x \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} h \\ k \end{bmatrix}\right)=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Image $(x_1, y_1) = \left(d_y(r_y x+h), d_x(r_x y+k)\right)$

$$y_1(x) = d_x\left[r_x y\left(\frac{1}{r_y d_y}(x-d_y h)\right)+k \right] \tag{1.9}$$

8. Other more complex transformations

e.g. kind of inverse function after translation, with dilation/reflection:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{-ab}\begin{bmatrix} 0 & -a \\ -b & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{b} \\ \frac{1}{a} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{b} \\ \frac{1}{a} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} \frac{y_1}{b} - h \\ \frac{x_1}{a} - k \end{bmatrix}$$

Image : $(x_1, y_1) = (a(y+k), b(x+h))$

$$y(x) = \frac{x_1}{a} - k$$

$$\Rightarrow \frac{x_1}{a} - k = y\left(\frac{y_1}{b} - h\right)$$

Then solve this equation for a given y to find the new function $y_1(x_1)$

e.g. $y = 2x + 5$, $a = -3, b = 2, h = 1, k = 2$

$$\frac{x_1}{-3} - 2 = 2\left(\frac{y_1}{2} - 1\right) + 5$$

$$\frac{x_1}{-3} - 2 = y_1 + 3$$

$$y_1 = -\frac{x_1}{3} - 5$$

dropping the "1" in x_1,

$$y_1(x) = -\frac{x}{3} - 5$$

9. Counterclockwise rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

10. Counterclockwise rotation about x axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

11. Counterclockwise rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

12. Rotate about point (a,b) : first translate (-a,-b), do the rotation, then translate back:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$